Math 2101

Homework 1

Due: October 7, 2013

Problems marked with $*$ will be marked. The students are advised to work over ALL problems in the assignments, so as to keep pace with the material.

1. (a) Let z be a complex number satisfying $z^{10} + z^9 + \cdots + z + 1 = 0$. Show that

$$w = \frac{z^2+1}{z}$$

is a real number.

(b) Find the fourth roots of $16i$.

2. $*$ Let a and b be real numbers, z a complex number such that

$$a \cdot i^{-2012} + b \cdot i^{2010} = |2z+2|i^{2011} - |z-1|i^{-2013}.$$

Where are the images of z in the complex plane located?

3. Given the equation $z^2 - az + b = 0$ with $a, b \in \mathbb{R}$ and z_1 and z_2 its roots. We assume that $z_1 = 2 + i$.

   (a) Find a and b.

   (b) Show that $z_1^{2013} + z_2^{2013}$ is real and write an expression for it in terms of trigonometric numbers showing that it is a real number.

   (c) If A, B, C correspond to z_1, z_2 and z_3 in the plane with

$$z_3 = \frac{z_1}{z_2} + \frac{1}{5}(17+i),$$

   show that the triangle ABC is a right-isosceles triangle.

   (d) If $|w - z_1| = |\bar{w} - z_1|$ show that $w \in \mathbb{R}$.

4. $*$ Let $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$ with $a \neq b$ and $n \in N$ such that

$$(1+iz)^n = \frac{a+bi}{b+ai}.$$

   (a) Show that z is not a real number.

(b) Show that z lies on a circle. Find the radius and centre of the circle.

(c) Find the maximum modulus of such z.

(d) Show that $4 < |z - 3 + 4i| < 7$.

5. Prove the Lagrange identity

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leq i < j \leq n} |a_i \bar{b}_j - a_j \bar{b}_i|^2.$$

6. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1,$$

if $|a| < 1$ and $|b| < 1$.

7. $*$ Let z_1, z_2, z_3 be three complex numbers with $|z_j| = 2$ for $j = 1, 2, 3$. We are given that

$$\Re\left(\frac{z_1}{z_2}\right) = \Re\left(\frac{z_2}{z_3}\right) = \Re\left(\frac{z_3}{z_1}\right) = -\frac{1}{2}.$$

Show that $z_1 + z_2 + z_3 = 0$ and that the triangle with vertices z_1, z_2, z_3 is equilateral.

Hint: What happens when we rotate around 0 an equilateral triangle centered at 0? Why can we assume that z_1 is real and positive?

8. Express $\cos(5\phi)$ in terms of $\cos(\phi)$. Express $\sin(5\phi)$ in terms of $\sin(\phi)$.

9. In a triangle a median is the line joining a vertex with the midpoint of the opposite side. Show that the three medians intersect at the same point. This point is called the barycenter. If z_i are complex numbers corresponding to the vertices of the triangle, which complex number corresponds to the barycenter?

You must use complex numbers, not geometric arguments.

10. $*$ Prove that each of the following sets are open:

(a) $A = \{z : \Im(z) < 0\}$,

(b) $B = \{z : \Re(z) > 0 \text{ and } \Im(z) > 0\}$.

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Homework 2

Due: October 14, 2013

1. $*$ Verify the Cauchy-Riemann equations for

$$\text{(a) } f(z) = e^{-x}(\cos y - i \sin y), \quad \text{(b) } f(z) = \cos x \cosh y - i \sin x \sinh y.$$

Find $f'(z)$ in both cases.

2. $*$ Apply the definition of derivatives to show that if $f(z) = \Re(z)$, then $f'(z)$ does not exist anywhere.

3. Suppose that f is holomorphic in a region Ω. Prove that in any of the following cases

(a) $\Re(f)$ is constant;

(b) $\Im(f)$ is constant;

(c) f is real-valued;

(d) $\arg(f)$ is constant;

we can conclude that f is a constant. Do not use integration.

Hint for (d): Can you find a complex number c such that $g(z) = cf(z)$ is real-valued?

4. $*$ (a) Let

$$u(x, y) = \sinh(x) \cdot \sin(y).$$

Show that u is a harmonic function. Find all functions $v(x, y)$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic. Write f as a function of z.

Hint: Recall that

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \quad \cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

(b) For the harmonic function $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by the formula

$$v(x, y) = \frac{-y}{x^2 + y^2}$$

find all holomorphic functions $f(z)$ such that $\Im f(z) = v(x, y)$. Write f as a function of z.

5. Prove rigorously that, if the function $f(z)$ is holomorphic on $D(0, R)$, then $g(z) = \overline{f(\bar{z})}$ is also holomorphic on $D(0, R)$.

6. Recall the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Suppose that U and V are open sets in the complex plane. Assume that $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable in the real sense i.e. $w = f(x, y)$ has continuous partial derivatives in x and y, and $g(u, v)$ has continuous partial derivatives in u and v. We set $h = g \circ f$. Write $\partial g/\partial w$ and $\partial g/\partial \bar{w}$ as functions of $\partial g/\partial u$ and $\partial g/\partial v$. Show that the complex version of the chain rule is

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

This is an exercise in the chain rule from Methods 2.

7. $*$ Recall the polar coordinates in the plane:

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x),$$

which can be solved for x, y to give

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Show that in polar coordinates the Cauchy–Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the function $f(z) = 1/z$ is holomorphic when $z \neq 0$ and find its derivative.

8. $*$ Let $f(z) = u(x, y) + iv(x, y)$ be holomorphic on the domain Ω. Suppose that

$$u(x, y)^2 - u(x, y) \cdot v(x, y) + v(x, y)^2$$

is constant for all $z \in \Omega$. Show that f is a constant function.

Hint: Imitate the proof for $|f|$ constant.

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Homework 3

Due: October 21, 2013

The first four exercises of the homework deal with questions relating to lim sup and lim inf and can be solved without complex analysis.

1. $*$ Show that if $\langle a_n \rangle$ is a convergent sequence in $\mathbb{R}$ and $\lim a_n = l$, then $\limsup a_n = l$.

2. Let a_n, b_n be real and positive and assume that the sequence $\langle b_n \rangle$ is convergent with $\lim b_n = b > 0$. Assume also $\limsup a_n = a$. Show

$$\limsup(a_n b_n) = ab.$$

   Deduce that $\limsup \sqrt[n]{n|a_n|} = \limsup \sqrt[n]{|a_n|}$.

3. Show that, $\langle a_n \rangle_{n=0}^{\infty}$ is a sequence of non-zero complex numbers, then

$$\liminf \frac{|a_{n+1}|}{|a_n|} \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \frac{|a_{n+1}|}{|a_n|}.$$

4. $*$ Show that if $\langle a_n \rangle_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

   then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

   This is the ratio test and it can be used for the calculation of the radius of convergence of a power series.

5. Find the radius of convergence of the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

   Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$.

6. Show that

$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$$

   for $|z+1| < 1$.

7. $*$ Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss the convergence for $z = 1, -1, i$.

Hint: The n-th coefficient of this series is not $(-1)^n/n$.

8. Prove that, although all the following power series have radius of convergence $R = 1$,

(a) $\sum nz^n$ does not converge on any point of the unit circle $\{z, |z| = 1\}$,

(b) $\sum z^n/n^2$ converges at every point of the unit circle.

9. $*$ The Fibonacci numbers are defined by $f_0 = 1$, $f_1 = 1$,

$$f_n = f_{n-1} + f_{n-2}, \quad n = 2, 3, \ldots.$$

Define their generating function as

$$F(z) = \sum_{n=0}^{\infty} f_n z^n.$$

(a) Find a quadratic polynomial $Az^2 + Bz + C$ such that

$$(Az^2 + Bz + C)F(z) = 1.$$

(b) Use partial fractions to determine the following closed expression for f_n.

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

10. (i) Show that the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

can be calculated as

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

(ii) Show that for any analytic function $f(z)$ we have

$$\Delta|f(z)|^2 = 4|f'(z)|^2.$$

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Homework 4

Due: October 28, 2013

1. (a) Show that $e^{\bar{z}} = \overline{e^{z}}$.

   (b) Show that $e^{\bar{z}}$ is not holomorphic at any point in $\mathbb{C}$.

   (c) Find the image of the semi-infinite strip $x \geq 0$ and $0 \leq y \leq \pi$ under the transformation $w = e^z$. Exhibit corresponding portions of the boundaries.

2. $*$ For $z = x + iy$ show that

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

   Deduce that

$$|\cos(z)|^2 = \cos^2 x + \sinh^2(y),$$

   and

$$|\sinh(y)| \leq |\cos(z)| \leq \cosh(y).$$

3. $*$ Solve the equation $\cosh(z) = 1/2$.

4. Denote by Log (z) the principal logarithm. Show that if $\Re(z_1) > 0$ and $\Re(z_2) > 0$, then

$$\text{Log } (z_1 z_2) = \text{Log } (z_1) + \text{Log } (z_2).$$

   Show that this formula is not true in general by founding a counterexample.

5. Show that the transformation $w = z^2$ maps the lines $x = c$ for $c \neq 0$ onto the parabolas $v^2 = -4c^2(u - c^2)$ and the lines $y = d$ for $d \neq 0$ onto the parabolas $v^2 = 4d^2(u + d^2)$. Prove that at a point of intersection the parabolas meet orthogonally in two ways: (a) directly and (b) quoting a theorem.

6. $*$ Show that if $c_1 < 0$, the image of the half plane with equation $x < c_1$ under the transformation $w = 1/z$ is an open disc. Write the equation of the circle bounding this disc.

7. Let $w = \dfrac{z-1}{z+1}$. Show that it maps the right-half plane $\{z, \Re(z) > 0\}$ conformally onto the unit disc $\{z, |z| < 1\}$. Notice that the map must be bijective.

8. $*$ (a) Show that the map $w = \dfrac{z-i}{z+i}$ maps the upper half-plane $\{z; \Im(z) > 0\}$ conformally onto the unit disc $\{z; |z| < 1\}$.

(b) Find a conformal map from $\{z, \Re(z) > 0$ and $\Im(z) > 0\}$ onto the unit disc $\{z; |z| < 1\}$. *Hint*: To start use z^2.

9. Show that any transformation of the form

$$w = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 \cdot z}, \quad \theta \in \mathbb{R}, \quad |z_0| < 1$$

maps the disk $|z| \leq 1$ conformally into the disc $|w| \leq 1$.

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Homework 5

Due: November 11, 2013

1. $*$ Evaluate the integrals

$$I_j = \int_{\gamma_j} (\bar{z})^2 \, dz, \quad j = 1, 2, 3,$$

where

- γ_1 is the segment from 0 to $1+i$,
- γ_2 is the circular arc from $1+i$ to $i\sqrt{2}$ on the circle $|z| = \sqrt{2}$, traversed anticlockwise,
- γ_3 is the segment from $i\sqrt{2}$ to 0.

Hence determine the value of the integral

$$\int_{\gamma_1+\gamma_2+\gamma_3} (\bar{z})^2 \, dz.$$

2. $*$ Compute the integral

$$\int_{|z|=r} y \, dz$$

for the circle traversed anticlockwise, in two ways: first by using a parametrisation, and second, by observing that $y = (1/(2i))(z - \bar{z}) = (1/(2i))(z - r^2/z)$ on the circle.

3. In the following you are not allowed to use the residue theorem or Cauchy's integral formula, as they have not been discussed yet.

(a) Evaluate the integrals

$$\int_\gamma z^n \, dz$$

for all integers n, where γ is any circle not containing the origin, tranversed in the positive sense.

(b) Let a, b, r be numbers with $0 < a < r < b$ and let γ be the circle of radius r centered at the origin with positive orientation. Show that

$$\int_\gamma \frac{1}{(z-a)(z-b)} \, dz = \frac{2\pi i}{a-b}.$$

(c) Compute the integral

$$\int_{|z|=2} \frac{dz}{z^2-1}$$

for the positive sense of the circle. *Hint*: Find a primitive function of the integrand.

4. Map conformally the region inside both circles $|z-1|<1$ and $|z+i|<1$ to the upper-half plane $\mathbb{H}=\{z\in\mathbb{C}, \Im(z)>0\}$.

   Hint: Use first the map $w=z^{-1}$ and at some later stage use $w=z^2$.

5. Let C be the boundary of the triangle with vertices at the points $z=0$, $z=3i$ and $z=-4$, traversed anticlockwise. Without evaluating the integral show that

$$\left|\int_C (e^z-\bar{z})\,dz\right| \le 60.$$

6. $*$ (a) By considering the contour integral

$$\int_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z}$$

   prove that

$$\int_0^{2\pi} \cos^{2n} t\,dt = 2\pi \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

   (b) Prove that

$$\int_0^{\pi/2} \sin^{2n} t\,dt = \frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

7. $*$ Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)\,dx, \qquad \int_0^\infty e^{-ax}\sin(bx)\,dx, \qquad a,b>0$$

   by integrating the holomorphic function e^{-Az}, $A=\sqrt{a^2+b^2}$, over a sector with angle ω satisfying

$$\cos\omega = \frac{a}{A}.$$

   Do not use integration by parts.

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Homework 6

Due: November 18, 2013

1. $*$ Let C be the boundary of the square with sides along the lines $x = \pm 2$ and $y = \pm 2$ traversed anticlockwise. Compute

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz, \quad \int_C \frac{\cosh z}{z^3}\,dz, \quad \int_C \frac{1}{(z^2+1)^2}\,dz.$$

Do not use the residue theorem.

2. $*$ By evaluating the integral

$$\frac{1}{2\pi i}\int_C \frac{1}{(z-a)(z-a^{-1})}\,dz$$

around the unit circle C, prove that, if $0 < a < 1$, we have

$$\int_0^{2\pi} \frac{1}{1+a^2-2a\cos t}\,dt = \frac{2\pi}{1-a^2}.$$

You are not allowed to use the residue theorem.

3. $*$ By considering the contour integral

$$\int_C \frac{e^{az}}{z}\,dz$$

around the unit circle C traversed anticlockwise, show that

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

4. Show that, when f is holomorphic on an open set U containing the simple closed contour C and its interior, and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

5. Let $f(z)$ be holomorphic on the unit disc and $f(0) = 1/2$. By working with

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z}$$

prove that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{it})\cos^2\frac{t}{2}\,dt = 1 + f'(0), \quad \frac{2}{\pi}\int_0^{2\pi} f(e^{it})\sin^2\frac{t}{2}\,dt = 1 - f'(0).$$

6. $*$ Let C be a circle enclosing the distinct points $z_1, z_2, \ldots, z_n$. Let

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

be a polynomial of degree n with roots at these points. Let $f(z)$ be holomorphic in a disc that includes C. Show that

$$P(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{p(w)}\frac{p(w) - p(z)}{w - z}dw$$

is a polynomial of degree $n - 1$, with the property

$$P(z_i) = f(z_i), \quad i = 1, 2, \ldots n.$$

The following exercise is more theoretical.

7. Let f is holomorphic on an open set U containing the closed disk D with boundary the circle C. Show the Cauchy formulae for higher derivatives, which was proved in class for $n = 1$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz, \quad \forall n \in \mathbb{N},$$

where C is traversed anticlockwise and z_0 is inside C.

Hint: Use induction. Imitate the proof from the class. It helps to define $A = z - z_0$, $B = z - z_0 - h$ and recall the identity

$$A^k - B^k = (A - B)(A^{k-1} + A^{k-2}B + A^{k-3}B^2 + \cdots + AB^{k-2} + B^{k-1}).$$

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Homework 7

Due: November 27, 2013

1. Let $f(z)$ be holomorphic in the region $|z| \leq R$ with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let the partial sum of the series be defined as

$$s_N(z) = \sum_{n=0}^{N} a_n z^n.$$

Show that for $|z| < R$ we have

$$s_N(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{w^{N+1} - z^{N+1}}{w - z} \frac{dw}{w^{N+1}}.$$

2. $*$ Let f be entire.

   (a) Show that, if e^f is bounded, then f is constant.

   (b) Assume that $\Im(f)$ is bounded below. Show that f is a constant function.

3. For each of the following functions determine the isolated singularities and their nature. For the poles, find the order of the pole, the principal part and residue at the pole.

$$\text{(a)}\ \frac{z^2}{1+z}, \quad \text{(b)}\ \frac{\sin z}{z}, \quad \text{(c)}\ \frac{\cos z}{z} \quad \text{(d)}\ \tanh z.$$

4. What is the value of the integral

$$\int_C \frac{1}{z^2+1}\, dz,$$

where C is (i) the circle $|z| = 2$ traversed anticlockwise, (ii) the circle $|z - i| = 1$ traversed anticlockwise?

5. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)}\ \int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}. \qquad \text{(b)}\ \int_0^{\infty} \frac{\cos(ax)}{x^2+1}\, dx = \frac{\pi}{2} e^{-a}, \quad a \geq 0.$$

6. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_0^\infty \frac{1}{x^6+1}\,dx = \frac{\pi}{3}, \quad \text{(b)} \int_{-\infty}^\infty \frac{\cos(x)}{(x^2+4)(x^2+16)}\,dx = \frac{\pi}{12}\left(\frac{e^{-2}}{2} - \frac{e^{-4}}{4}\right).$$

7. (a) Show that on the interval $(0, \pi/2]$ the function $\sin u/u$ is decreasing. Deduce that

$$\sin u \geq \frac{2u}{\pi}, \quad u \in [0, \pi/2].$$

(b) Use contour integration to show the value of the (Fresnel) integrals

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \sqrt{2\pi}/4.$$

Hint: Consider the function $f(z) = e^{iz^2}$ and a contour that includes the line segment from 0 to $Re^{i\pi/4}$ and the circular arc from R to $Re^{i\pi/4}$.

8. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Hint: Integrate $f(z) = e^{iz}/z$ in the indented semicircular contour.

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Homework 8

Due: December 4, 2013

1. $*$ Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_0^{2\pi} \frac{1}{3+\sin(t)}\,dt = \frac{\pi}{\sqrt{2}}, \quad \text{(b)} \int_0^{\infty} \frac{\log x}{x^2+1}\,dx = 0,$$

$$\text{(c)} \int_0^{2\pi} \frac{dt}{(a+\cos t)^2} = 2\pi a/(a^2-1)^{3/2} \quad (a>1).$$

2. Explain why the following integrals have the given value using the residue theorem. Complete explanations are required.

$$\text{(a)} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)\pi}{2\cdot 4\cdot 6\cdots(2n)},$$

$$\text{(b)} \int_0^{\pi/2} \frac{1}{a+\cos^2 t}\,dt = \frac{\pi}{2\sqrt{a^2+a}}, \quad a>0.$$

3. Let $p(z)$ and $q(z)$ be holomorphic in the disk $D(z_0, r)$, $r>0$. Assume that $q(z_0)=0$, while $q'(z_0)\neq 0$. Assume that $p(z_0)\neq 0$.

   (a) Explain why $f(z) = p(z)/q(z)$ has a simple pole at z_0.

   (b) Show that

$$\text{res }(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

   (c) What happens if, instead of $q'(z_0)\neq 0$, we assume that $q'(z_0)=0$, while $q''(z_0)\neq 0$? Show that in this case

$$\text{res }(f, z_0) = 2\frac{p'(z_0)}{q''(z_0)} - \frac{2}{3}\frac{p(z_0)q'''(z_0)}{[q''(z_0)]^2}.$$

   (d) Identify all the isolated singular points of $f(z) = \cot z$. Which ones are poles? Find the corresponding residues.

4. $*$ Show that there does not exist a holomorphic function f on $D(0,1)$ such that

$$f(1/n) = \begin{cases} 1+2/n, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}$$

Math 2101

Homework 9

NOT DUE

While this assignment will not be marked, please study the Part A for the final exam in May.

PART A

1. (a) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

   (b) Find the number of the roots of the equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

   in the annulus $\{z : 1 < |z| < 2\}$.

2. Let λ be real and $\lambda > 1$, Show that the equation

$$ze^{\lambda - z} = 1$$

   has exactly one solution in the disc $|z| = 1$, which is real and positive.

3. Let C be the unit circle $|z| = 1$ traversed anticlockwise. Determine the variation of the argument $\Delta_C \arg f(z)$ for the functions

$$\text{(a) } f(z) = z^2, \quad \text{(b) } f(z) = \frac{z^3 + 2}{z}.$$

4. Suppose that $f(z)$ is holomorphic in a punctured disc $D'(z_0, \delta)$. Suppose that for some constant M and for all $z \in D'(z_0, \delta)$ we have

$$|f(z)| \leq M|z - z_0|^{-1/4}.$$

   Show that the singularity of f at z_0 is removable.

PART B

This part contains a few harder problems that are interesting and aimed for advanced study.

1. (Schwarz's Lemma) Let f be holomorphic on the unit disc $\{z, |z| < 1\}$ with

   (a) $|f(z)| \leq 1$ for $|z| < 1$,

   (b) $f(0) = 0$.

   Show that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for $|z| < 1$. Moreover, if $|f'(0)| = 1$ or $|f(w)| = |w|$ for some point w with $0 < |w| < 1$, then we can find a c with $|c| = 1$ and
$$f(z) = cz, \quad \forall z \in \mathbb{C}, |z| < 1.$$

2. Prove that
$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)} \quad (u \notin \mathbb{Z}), \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$
   using the function $f(z) = \dfrac{\pi \cot(\pi z)}{(u+z)^2}$ integrated over the boundary of the square $[-(N+1/2), N+1/2] \times [-(N+1/2), N+1/2]$, $N \geq |u|$, $N \in \mathbb{N}$. This is one of the many derivations of the value $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, due originally to Euler.

3. In this problem $\int_{c-i\infty}^{c+i\infty}$ denotes a contour integral along the vertical line $\Re(s) = c$ traversed upwards.

   (a) Prove that for $c > 0$ we have $\displaystyle\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} ds = \begin{cases} \log x, & x > 1, \\ 0, & 0 < x \leq 1. \end{cases}$

   (b) Prove that, for $c > 0$, $\displaystyle\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1, & x > 1, \\ 1/2, & x = 1, \\ 0, & 0 < x < 1. \end{cases}$ (Perron formula)

   (c) Let the function $f(s)$ be defined by the absolutely convergent series
$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) > a \geq 0.$$
   Show that for $x \notin \mathbb{Z}$
$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} ds, \quad c > a.$$